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SOME EXACT SOLUTIONS FOR THE BENDING OF BEAMS WITH SPATIALLY STOCHASTIC STIFFNESS

I. ELISHAKOFF, Y. J. REN
Center for Applied Stochastic Research and Department of Mechanical Engineering Florida Atlantic University, Boca Raton, FL 33431-0991, U.S.A.

and

M. SHINOZUKA
Department of Civil Engineering and Operational Research Princeton University, Princeton, NJ 08544, U.S.A.

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Abstract—To the best of the knowledge of the authors, there are no exact solutions available for the bending of beams with spatially stochastic stiffness. Investigators are therefore utilizing various approximate techniques. In the present study, a new method is developed to obtain exact solutions for first and second moments of displacements for statically determinate beams that have spatially random stiffness. The method is based on the full probabilistic characterization of the random stiffness so that the solutions are valid for any value of the coefficient of variation of the stiffness. The deterministic governing equations and boundary conditions derived for both first and second moments allow, apparently for the first time in the literature, the exact solutions for the mean and covariance functions of the displacements to be determined. Two governing equations are uncoupled from each other and can be solved separately. Several exact solutions for the mean and covariance functions are obtained to illustrate the application of the method. It is hoped that the exact solutions determined will serve as benchmark solutions to enable the researchers to check the accuracy of various approximate analytical and numerical techniques on the test solutions presented in this study.

INTRODUCTION

Structures involving spatially random material and/or geometrical parameters are referred to as *stochastic structures*. The analysis of stochastic structures has attracted the significant interest of many researchers in recent decades. However, difficulties arise in obtaining exact solutions of stochastic structures since the appropriate governing equations constitute random differential equations with random coefficient functions, and possibly with random boundary conditions. Several approximate methods (both analytical and numerical) have therefore been developed to address the problem. Most of these methods are based on a perturbation technique or a series expansion method and are applicable only to small coefficients of variation of the random parameters. Analytical perturbation methods include those of Molyneux and Beran (1965) and Lomakin (1970), whereas perturbation-based numerical methods are represented by the stochastic finite element methods developed by Nakagiri and Hisada (1981, 1985), Yamazaki, Shinozuka, and Dasgupta (1988), and others.

For beam-bending problems, both a spatially random material parameter (Young's modulus) and geometrical parameters (dimensions of the cross-section) can be combined into one parameter (the bending stiffness). The governing equation of the beam bending is a differential equation with spatially varying random coefficient, along with random boundary conditions. In this paper, we derive exact expressions for the first and second moments of the displacement. The solutions for the mean and covariance function of the beam displacement with spatially random stiffness are precisely obtained if the inner forces are statically determinate. The deterministic governing equations and boundary conditions for the first and second moments (mean and covariance function) of displacements are shown to be uncoupled from each other, on the basis of full probabilistic knowledge of the random stiffness. Two deterministic governing equations can be solved separately by means of deterministic methods, analytically or numerically.

It must be stressed that the search for closed-form solutions is desirable, since, by this means, the influence of various parameters would be directly "visible" without the parametric analysis necessary in numerical solutions. Moreover, a closed-form solution serves as a benchmark solution for comparison purposes, namely, in the cases where the exact solution is unavailable and numerical methods (for example, the method of stochastic finite elements) should be resorted to. The computer program developed can be checked for the special cases in which a closed-form solution is present, and the latter will serve as a test problem. The solutions derived in this paper are applicable to any value of the coefficient of variation of the random stiffness. The proposed solutions can therefore be utilized to verify the accuracy of existing perturbation solutions.

Three beam problems with spatially random stiffness are examplified to show the application of the present method. They are: (i) a cantilever beam under an end moment, with the spatially random stiffness possessing a uniform distribution and a second-order auto-regressive correlation; (ii) a cantilever beam under a linearly distributed force, with uniformly distributed random stiffness possessing triangular correlation; and (iii) a simply supported beam under uniform force, with the random stiffness of the beam having triangular distribution with exponential correlation. Exact solutions for the mean and covariance functions of displacements for these problems are obtained. It is worthy of note that the displacement function is no longer a homogeneous random field even if the random stiffness is homogeneous, owing to the existence of boundaries.

BASIC EQUATIONS

The beam-bending problem with spatially stochastic stiffness is governed by the following equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[EI(x)\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right] = q(x) \tag{1}$$

where $w(x)$ = the displacement, $q(x)$ = the transverse distributed force, and $EI(x)$ = the bending stiffness. which is assumed to be a spatially random field. Equation (1) can be rewritten as

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \frac{m(x)}{EI(x)} \tag{2}$$

where

$$m(x) = -\int_0^x \int_0^v q(u)\,\mathrm{d}u\,\mathrm{d}v + Q_0 x + M_0 \tag{3}$$

is the bending moment in the cross-section of the beam, and M_0 and Q_0 are constants of integration representing the bending moment and shear force at the end $x = 0$, respectively. Assume that moments and shear forces in the beam are statically determinate, namely, that M_0 and Q_0 are independent of the stochastic stiffness but dependent upon the loading and boundary conditions. By taking expectation from eqn (2), we obtain

$$\frac{\mathrm{d}^2 \bar{w}}{\mathrm{d}x^2} = \frac{m(x)}{D_0(x)} \tag{4}$$

where $\bar{w}(x) = \mathrm{E}[w(x)]$ is the mean of the displacement $w(x)$, and $D_0(x)$ is defined by

$$\frac{1}{D_0(x)} = \mathrm{E}\left[\frac{1}{EI(x)}\right] \tag{5}$$

where E(·) signifies mathematical expectation. Pre-multipying eqn (4) by $D_0(x)$ and differentiating the result twice, we obtain the governing equation for the mean value of displacement response $\bar{w}(x)$:

$$\frac{d^2}{dx^2}\left[D_0(x)\frac{d^2\bar{w}}{dx^2}\right] = q(x). \tag{6}$$

Equation (6) is the governing equation for the mean of displacement. In form, eqn (6) is identical to the equation of the beam with deterministic stiffness $D_0(x)$. Subtracting eqn (4) from eqn (2) and multiplying the resulting equation by itself but evaluated at the cross-section y, we obtain

$$\frac{d^2[w(x)-\bar{w}(x)]}{dx^2}\frac{d^2[w(y)-\bar{w}(y)]}{dy^2} = m(x)m(y)\left[\frac{1}{EI(x)}-\frac{1}{D_0(x)}\right]\left[\frac{1}{EI(y)}-\frac{1}{D_0(y)}\right]. \tag{7}$$

Taking expectation of eqn (7) and then partially differentiating the result twice with respect to x and twice with respect to y, we arrive at the governing equation for the covariance function of displacement:

$$\frac{\partial^4}{\partial x^2\,\partial y^2}\left[D_1(x,y)\frac{\partial^4 C(x,y)}{\partial x^2\,\partial y^2}\right] = q(x)q(y) \tag{8}$$

where $C(x,y) = \mathrm{E}\{[w(x)-\bar{w}(x)][w(y)-\bar{w}(y)]\}$ is the covariance function of displacements $w(x)$ at position x and $w(y)$ at position y, and

$$\frac{1}{D_1(x,y)} = \mathrm{E}\left\{\left[\frac{1}{EI(x)}-\frac{1}{D_0(x)}\right]\left[\frac{1}{EI(y)}-\frac{1}{D_0(y)}\right]\right\}. \tag{9}$$

The associated boundary conditions are proved in the Appendix. For the mean displacement $\bar{w}(x)$, the boundary conditions at $x = 0$ and $x = L$ read:

$$\bar{w} = 0 \quad \text{or} \quad \frac{d}{dx}\left[D_0(x)\frac{d^2\bar{w}}{dx^2}\right] = \bar{Q} \tag{10}$$

and

$$\frac{d\bar{w}}{dx} = 0 \quad \text{or} \quad D_0\frac{d^2\bar{w}}{dx^2} = \bar{M} \tag{11}$$

where $\bar{M}$ and $\bar{Q}$ are the prescribed moment and shear force at the ends, respectively. The boundary conditions for the covariance function $C(x,y)$ are

$$\frac{\partial C}{\partial x} = 0 \qquad \text{or} \quad D_1 \frac{\partial^4 C}{\partial x^2\, \partial y^2} = \bar{M}m(y)$$

$$C = 0 \quad \text{or} \quad \frac{\partial}{\partial x}\left[D_1 \frac{\partial^4 C}{\partial x^2\, \partial y^2}\right] = \bar{Q}m(y) \tag{12}$$

at $x = 0$ and $x = L$, and

$$\frac{\partial C}{\partial y} = 0 \qquad \text{or} \quad D_0 \frac{\partial^4 C}{\partial x^2\, \partial y^2} = \bar{M}m(x)$$

$$C = 0 \quad \text{or} \quad \frac{\partial}{\partial y}\left[D_1 \frac{\partial^4 C}{\partial x^2\, \partial y^2}\right] = \bar{Q}m(x) \tag{13}$$

at $y = 0$ and $y = L$.

In order to compare later the exact solutions obtained in the present paper and the perturbation solutions, we hereby first briefly outline the basic equations based on the perturbation technique.

PERTURBATION METHOD

Let the bending stiffness $EI(x)$ and the displacement $w(x)$ be represented as follows, respectively:

$$\left.\begin{aligned} EI(x) &= EI_0[1+\alpha(x)] \\ w(x) &= w_0(x)+w_1(x)+w_2(x)+\cdots \end{aligned}\right\} \tag{14}$$

where EI_0 is the mean stiffness and $\alpha(x)$ is the normalized random field of $EI(x)$. The parameter $\alpha(x)$ can be considered as a perturbation of $EI(x)$ from the mean EI_0; it is small in the sense that $|\alpha(x)| \ll 1$. The parameter $w_0(x)$ is the displacement corresponding to the mean stiffness EI_0, and $w_i(x)$ is the ith order perturbation of $w(x)$. Substituting eqn (14) into eqn (2), we obtain

$$EI_0[1+\alpha(x)]\frac{\mathrm{d}^2}{\mathrm{d}x^2}[w_0(x)+w_1(x)+w_2(x)+\cdots] = m(x). \tag{15}$$

Based on the perturbation technique, eqn (15) gives

$$\frac{\mathrm{d}^2 w_0(x)}{\mathrm{d}x^2} = \frac{m(x)}{EI_0}$$

$$\frac{\mathrm{d}^2 w_1(x)}{\mathrm{d}x^2} = -\alpha(x)\frac{\mathrm{d}^2 w_0(x)}{\mathrm{d}x^2}$$

$$\frac{\mathrm{d}^2 w_{i-1}(x)}{\mathrm{d}x^2} = -\alpha(x)\frac{\mathrm{d}^2 w_i(x)}{\mathrm{d}x^2}, \quad i = 1, 2, \cdots. \tag{16}$$

Thus, within the first-order perturbation, the mean $\bar{w}^{(I)}(x)$ and covariance function $C^{(I)}(x, y)$ of displacement are governed, respectively, by

$$\left.\begin{aligned}\frac{d^2\bar{w}^{(I)}(x)}{dx^2} &= \frac{m(x)}{EI_0}\\ \frac{\partial^4 C^{(I)}(x,y)}{\partial x^2 \partial y^2} &= \frac{m(x)m(y)}{(EI_0)^2}C_{\alpha\alpha}(x,y)\end{aligned}\right. \tag{17}$$

where $C_{\alpha\alpha}(x,y) = \mathrm{Cov}\,[\alpha(x), \alpha(y)]$. Within the second-order perturbation, the mean $\bar{w}^{(II)}(x)$ and correlation function $C^{(II)}(x,y)$ are governed, respectively, by the following equations:

$$\left.\begin{aligned}\frac{d^2\bar{w}^{(II)}(x)}{dx^2} &= [1+\sigma^2(x)]\frac{m(x)}{EI_0}\\ \frac{\partial^4 C^{(II)}(x,y)}{\partial x^2 \partial y^2} &= \frac{m(x)m(y)}{(EI_0)^2}[C_{\alpha\alpha}(x,y) - C_{\alpha^2\alpha}(x,y) - C_{\alpha\alpha^2}(x,y) + C_{\alpha^2\alpha^2}(x,y) - \sigma^2(x)\sigma^2(y)]\end{aligned}\right\} \tag{18}$$

where $C_{\alpha^r\alpha^s}(x,y) = \mathrm{Cov}\,[\alpha^r(x), \alpha^s(y)]$ and $\sigma^2(x) = C_{\alpha\alpha}(x,x)$.

EXACT SOLUTION FOR CANTILEVER BEAM UNDER LINEARLY DISTRIBUTED LOAD

Consider first a clamped-free beam subjected to linearly distributed load $q(x) = q_0 + q_1 x/L$. The beam is free at $x = 0$ and clamped at $x = L$. The stiffness of the beam $EI(x)$ is assumed to be a spatially random field. Let $EI(x) = EI_0[1+\alpha(x)]$, and suppose that the normalized random field $\alpha(x)$ possesses a jointly uniform distribution with the following probabilistic density

$$f_{\alpha(x,y)}(u,v) = \frac{1}{4a^2}\left[1 + \frac{3\rho(x,y)}{a^2}uv\right], \quad u, v \in [-a, a] \tag{19}$$

where a = constant and $\rho(x,y)$ = the coefficient of correlation, which is assumed to be triangular, given by

$$\rho(x,y) = 1 - \frac{|x-y|}{L}, \quad |x-y| < L. \tag{20}$$

The governing equations for the mean value $\bar{w}$ and covariance function $C(x,y)$ are, respectively:

$$D_0\frac{d^4\bar{w}}{dx^4} = q_0 + \frac{q_1 x}{L} \tag{21}$$

$$\frac{\partial^4}{\partial x^2 \partial y^2}\left[D_1(x,y)\frac{\partial^4 C(x,y)}{\partial x^2 \partial y^2}\right] = \left(q_0 + \frac{q_1 x}{L}\right)\left(q_0 + \frac{q_1 y}{L}\right) \tag{22}$$

where

$$\frac{1}{D_0} = \frac{1}{2aEI_0}\ln\frac{1+a}{1-a} \tag{23}$$

and

$$\frac{1}{D_1(x,y)} = \frac{\rho(x,y)}{\bar{D}_{1u}}, \quad \frac{1}{\bar{D}_{1u}} = \frac{3}{(2a^2 EI_0)^2}\left(2a - \ln\frac{1+a}{1-a}\right)^2. \tag{24}$$

Boundary conditions in eqns (10–13) are simplified to

$$\left.\begin{aligned} \frac{d^2\bar{w}}{dx^2} = 0 \quad \frac{d^3\bar{w}}{dx^3} = 0 \quad \text{at} \quad x = 0, \\ \bar{w} = 0, \quad \frac{d\bar{w}}{dx} = 0 \quad \text{at} \quad x = L \end{aligned}\right\} \tag{25}$$

and

$$\left.\begin{aligned} \frac{\partial^2 C}{\partial x^2} = 0, \quad \frac{\partial^3 C}{\partial x^3} = 0 \quad \text{at} \quad x = 0, \\ C = 0, \quad \frac{\partial C}{\partial x} = 0 \quad \text{at} \quad x = L \end{aligned}\right\} \tag{26}$$

$$\begin{aligned} \frac{\partial^2 C}{\partial y^2} = 0, \quad \frac{\partial^3 C}{\partial y^3} = 0 \quad \text{at} \quad y = 0, \\ C = 0, \quad \frac{\partial C}{\partial y} = 0 \quad \text{at} \quad y = L. \end{aligned} \tag{27}$$

The solution for the mean displacement $\bar{w}(x)$ is straightforward and is obtained by integrating eqn (21) four times and satisfying the boundary conditions, eqn (25). The result reads:

$$\bar{w}(x) = \frac{q_0}{24D_0}(x^4 - 4L^3x + 3L^4) + \frac{q_1}{120D_0L}(x^5 - 5L^4x + 4L^5). \tag{28}$$

It is seen that the mean displacement $\bar{w}$ coincides with that of a beam that has a deterministic stiffness D_0. Expanding $1/D_0$ in eqn (28) with respect to the parameter a yields

$$\bar{w}(x) = w_0(x)\left(1 + \frac{a^2}{3} + \frac{a^4}{5} + \frac{a^6}{7} + \cdots\right) \tag{29}$$

where

$$w_0(x) = \frac{q_0}{24EI_0}(x^4 - 4L^3x + 3L^4) + \frac{q_1}{120EI_0L}(x^5 - 5L^4x + 4L^5) \tag{30}$$

is the displacement of the same beam with deterministic stiffness EI_0. It will be shown later that the first and second terms of eqn (29) coincide with the first- and second-order perturbation solutions, respectively.

To obtain the solution of the covariance function $C(x, y)$ from eqn (22) and the boundary conditions of eqns (26, 27), we first integrate eqn (22) with respect to x twice and with respect to y twice and reach

$$\frac{\partial^4 C(x,y)}{\partial x^2 \partial y^2} = \frac{1}{4\bar{D}_{1u}}\left(q_0 x^2 + \frac{q_1 x^3}{3L}\right)\left(q_0 y^2 + \frac{q_1 y^3}{3L}\right)\left(1 - \frac{|x-y|}{L}\right) \tag{31}$$

under the free boundary conditions at $x = 0$ and $y = 0$. The solution of eqn (31) is composed of a complementary solution $\psi(x,y)$ and a particular solution $\phi(x,y)$. The complementary solution $\psi(x,y)$ can be written as follows:

$$\psi(x,y) = \mathrm{f}_1(x) + y\mathrm{f}_2(x) + \mathrm{g}_1(y) + x\mathrm{g}_2(y) \tag{32}$$

where $\mathrm{f}_1(x)$, $\mathrm{f}_2(x)$, $\mathrm{g}_1(y)$, and $\mathrm{g}_2(y)$ are four arbitrary functions. The particular solution of eqn (31) reads:

$$\begin{aligned}\phi(x,y) = \frac{y^4}{2880L\bar{D}_{1u}}\bigg\{&\frac{q_0^2}{7}(35Lx^4 - 21x^5 + 21yx^4 - 15xy^4 + 7y^5)\\ &+ \frac{q_0 q_1}{9L}(9Lx^5 - 6x^6 + 9Lyx^4 - 6y^2x^4 - 8xy^5 + 4y^6)\\ &+ \frac{yq_1^2}{495L^2}(99Lx^5 - 66x^6 + 66yx^5 - 44xy^5 + 24y^6)\bigg\}, \quad \text{for} \quad x \geqslant y.\end{aligned} \tag{33}$$

The particular solution for $x \leqslant y$ can be obtained from eqn (33) by formal replacement of x by y and y by x, owing to symmetry in x and y. The boundary conditions at the fixed end $x = L$ and $y = L$ require

$$\left.\begin{aligned}\mathrm{f}_1(L) + y\mathrm{f}_2(L) + \mathrm{g}_1(y) + L\mathrm{g}_2(y) + \phi(L,y) &= 0\\ \mathrm{f}_1'(L) + y\mathrm{f}_2'(\mathrm{L}) + \mathrm{g}_2(y) + \phi_1(L,y) &= 0\\ \mathrm{f}_1(x) + L\mathrm{f}_2(x) + \mathrm{g}_1(L) + x\mathrm{g}_2(L) + \phi(L,x) &= 0\\ \mathrm{f}_2(x) + \mathrm{g}_1'(L) + x\mathrm{g}_2'(L) + \phi_1(L,x) &= 0\end{aligned}\right\}. \tag{34}$$

Solving out functions $\mathrm{f}_1(x)$, $\mathrm{f}_2(x)$, $\mathrm{g}_1(y)$, and $\mathrm{g}_2(x)$ from the above conditions, we obtain

$$\left.\begin{aligned}\mathrm{f}_2(x) &= -\phi_1(L,x) - \mathrm{g}_1'(L) - x\mathrm{g}_2'(L)\\ \mathrm{f}_1(x) &= -\phi(L,x) + \phi_1(L,x) - \mathrm{g}_1(L) - x\mathrm{g}_2(L) + L\mathrm{g}_1'(L) + xL\mathrm{g}_2(L)\\ \mathrm{g}_2(y) &= \phi_1(L,L) - \phi_1(L,y) + (L-y)\phi_{12}(L,L) + \mathrm{g}_2(L) - (L-y)\mathrm{g}_2'(L)\\ \mathrm{g}_1(y) &= \phi(L,L) - \phi(L,y) - (L-y)\phi_1(L,L) + \mathrm{g}_1(L) + L\mathrm{g}_2(L) - L\mathrm{g}_2(y)\\ &\quad - (L-y)\mathrm{g}_1'(L) - L(L-y)\mathrm{g}_2'(L)\end{aligned}\right\}. \tag{35}$$

By substituting eqn (35) back into eqn (32) and combining the complementary and particular solutions, we obtain the covariance function $C(x,y)$:

$$\begin{aligned}C(x,y) = {}&\phi(x,y) - \phi(L,x) - \phi(L,y) + (L-y)\phi_1(L,x) + (L-x)\phi_1(L,y) + \phi(L,L)\\ &+ (L-x)(L-y)\phi_{12}(L,L) - (L-x)\phi_1(L,L) - (L-y)\phi_1(L,L), \quad \text{for} \quad x \geqslant y\end{aligned} \tag{36}$$

$$\begin{aligned}C(x,y) = {}&\phi(y,x) - \phi(L,x) - \phi(L,y) + (L-y)\phi_1(L,x) + (L-x)\phi_1(L,y) + \phi(L,L)\\ &+ (L-x)(L-y)\phi_{12}(L,L) - (L-x)\phi_1(L,L) - (L-y)\phi_1(L,L), \quad \text{for} \quad x \leqslant y\end{aligned} \tag{37}$$

where

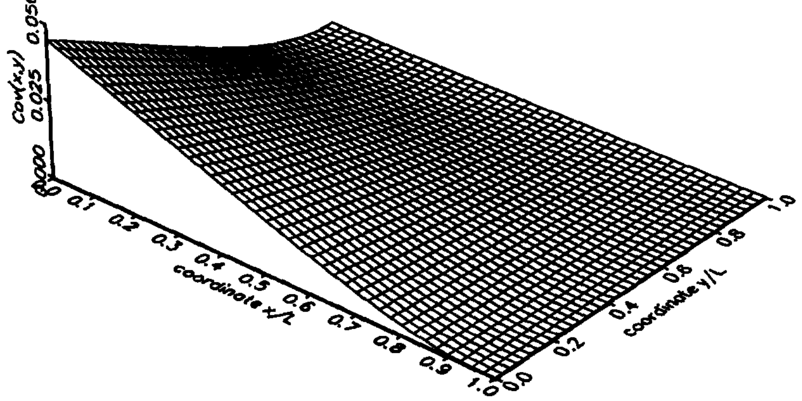

Fig. 1. Covariance function $C(x, y)$ of the displacement for cantilever beam subjected to linearly distributed load, normalized by $q_1^2/36L^2\bar{D}_{1u}$.

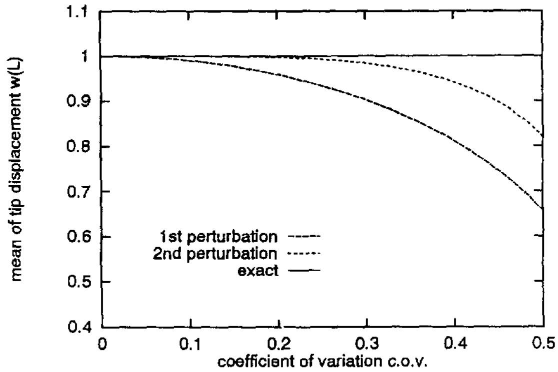

Fig. 2. Comparison of means of the tip displacement obtained by perturbation methods and the exact solution.

$$\left.\begin{aligned} \phi_1(u,v) &= \frac{\partial\phi(u,v)}{\partial u}, \quad \phi_2(u,v) = \frac{\partial\phi(u,v)}{\partial v} \\ \phi_{12}(u,v) &= \frac{\partial^2\phi(u,v)}{\partial u\partial v} \end{aligned}\right\}. \tag{38}$$

It can be proved that $\phi_1(L, L) = \phi_2(L, L)$. The perturbation solutions are obtained as

$$\left.\begin{aligned} \bar{w}^{(I)}(x) &= w_0(x), \quad \bar{w}^{(II)}(x) = w_0(x)\left(1+\frac{a^2}{3}\right) \\ C^{(I)}(x,y) &= C^{(II)}(x,y) = \frac{a^4}{9}\left[\left(1-\frac{1}{2a}\ln\frac{1+a}{1-a}\right)^2\right]^{-1} C(x,y) \end{aligned}\right\}. \tag{39}$$

Figure 1 portrays the correlation function $C(x, y)$ between any pair of points x and y for the particular case $q_0 = 0$, normalized by the constant $q_1^2/36L^2\bar{D}_{1u}$. It may be seen that $C(x, y)$ reaches its extreme value at $x = y$ for any fixed y, as it should be, since $C(y, y)$ represents the variance of the displacement; $C(x, y)$ attains its maximum variance at $x = y = 0$. Figures 2 and 3 illustrate the variation of the first and second perturbation solutions of the mean and variance of tip displacement versus the coefficient of variation of the random stiffness, normalized by the exact solutions. It may be seen, as expected, that only when the coefficient of variation is small are the perturbation solutions acceptable.

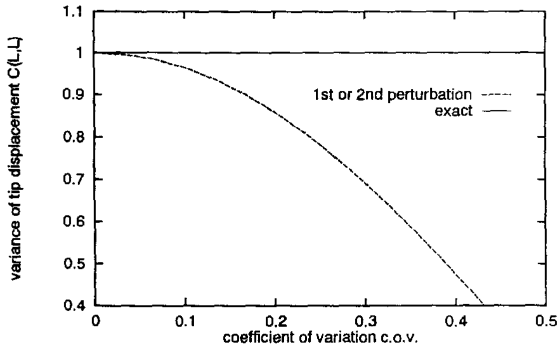

Fig. 3. Comparison of variances of the tip displacement obtained by perturbation methods and the exact solution.

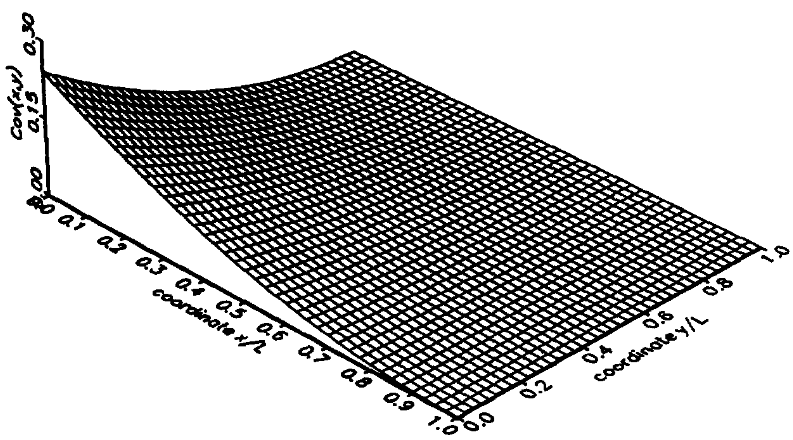

Fig. 4. Covariance function $C(x, y)$ of the displacement for a cantilever beam subjected to an end moment, normalized by $M^2/\bar{D}_{1u}$.

EXACT SOLUTION FOR CANTILEVER BEAM UNDER END MOMENT

Consider the same beam as in Example 1 but subjected to end moment M. The spatially random stiffness $EI(x) = EI_0[1+\alpha(x)]$ is assumed to possess jointly a uniform distribution and a so-called second-order auto-regressive correlation

$$\rho(x,y) = \left[1+\frac{|x-y|}{L}\right]e^{-|x-y|/L}. \tag{40}$$

The mean displacement $\bar{w}(x)$ reads:

$$\bar{w}(x) = \frac{m^2}{2\bar{D}_0}(x-L)^2. \tag{41}$$

The covariance function $C(x, y)$ can be solved in a similar way to the method used in the previously considered beam subjected to a linearly distributed force. The complementary solution $\psi(x, y)$ and the boundary conditions are the same as those in eqn (32) and eqn (34), respectively. The particular solution $\phi(x, y)$ is now obtained as

$$\begin{aligned}\phi(u,v) = \frac{M^2}{\bar{D}_{1u}}[&5L^4+4L^3x-4L^3y-3L^2xy+2Lxy^2-\tfrac{2}{3}Ly^3 \\ &-(5L^4+L^3x+4L^3y+L^2xy)\,e^{-x/L}-(5L^4+L^3y+4L^3x+L^2xy)\,e^{-y/L} \\ &+L^3(5L+x-y)\,e^{-(x-y)/L}].\end{aligned} \tag{42}$$

The covariance function $C(x, y)$ has the same final expressions as those given in eqns (36,37). Figure 4 shows the variation of $C(x, y)$ with positions, normalized by the constant

$M^2/\bar{D}_{1u}$. Again, as expected, $C(x, y)$ reaches its extremum at $x = y$ for a fixed y, and its maximum value appears at $x = y = 0$.

EXACT SOLUTION FOR SIMPLY SUPPORTED BEAM SUBJECTED TO UNIFORMLY DISTRIBUTED LOAD

Consider now a simply supported beam subjected to a uniformly distributed load q. The stiffness of the beam $EI(x) = EI_0[1+\alpha(x)]$ is assumed to be a spatially homogeneous random field. The normalized random field $\alpha(x)$ is assumed to satisfy a joint triangular distribution with density function (Gumbel, 1960):

$$\left.\begin{array}{c} \mathrm{f}_{\alpha(x,y)}(u, v) = \mathrm{f}(u)\mathrm{f}(v)\{1+\frac{150}{49}\rho(x, y)[2\mathrm{F}(u)-1][2\mathrm{F}(v)-1]\} \\ u, v \in [-b, b] \end{array}\right\} \tag{43}$$

where b = constant and $\mathrm{f}(t)$ and $\mathrm{F}(t)$ are the marginal triangular probabilistic density function and distribution function, respectively, namely:

$$\begin{aligned} \mathrm{f}(t) &= \frac{1}{b}\left(1-\frac{|t|}{b}\right), \quad t \in [-b, b] \\ \mathrm{F}(t) &= \frac{1}{2}+\frac{t}{b}-\frac{t|t|}{2b^2}, \quad t \in [-b, b]. \end{aligned} \tag{44}$$

The correlation structure $\rho(x, y)$ is assumed to be exponential:

$$\rho(x, y) = \mathrm{e}^{-|x-y|/L}. \tag{45}$$

By definitions in eqns (5) and (9), we have

$$\frac{1}{D_0} = \frac{1}{EI_0 b}\left[\ln\left(\frac{1+b}{1-b}\right)-\frac{1}{b}\ln\left(\frac{1}{1-b^2}\right)\right] \tag{46}$$

and

$$\left.\begin{array}{l} \dfrac{1}{D_1(x, y)} = \dfrac{1}{\bar{D}_{1t}}\rho(x, y) \\ \dfrac{1}{\bar{D}_{1t}} = \dfrac{150}{49b^2(EI_0)^2}\left[\dfrac{5}{3}+\dfrac{2}{b^2}-\dfrac{1+2b^2}{b^3}\ln\dfrac{1+b}{1-b}+\dfrac{3}{b^2}\ln\dfrac{1}{1b^2}\right]^2 \end{array}\right\}. \tag{47}$$

As previously mentioned, the mean displacement is identical to the displacement of a beam with deterministic stiffness D_0. Hence we have

$$\bar{w}(x) = \frac{qx}{24\bar{D}_0}(L^3-2Lx^2+x^3). \tag{48}$$

The covariance function $C(x, y)$ of the displacement is governed by

$$\frac{\partial^4}{\partial x^2 \partial y^2}\left[D_1(x, y)\frac{\partial^4 C(x, y)}{\partial x^2 \partial y^2}\right] = q^2 \tag{49}$$

with attendant boundary conditions

$$C = 0, \quad \frac{\partial^2 C}{\partial x^2} = 0 \quad \text{at} \quad x = 0, L \tag{50}$$

$$C = 0, \quad \frac{\partial^2 C}{\partial y^2} = 0 \quad \text{at} \quad y = 0, L. \tag{51}$$

Eqn (49) can be reduced to

$$\frac{\partial^4 C(x,y)}{\partial x^2 \partial y^2} = \frac{q^2}{4\bar{D}_{1t}}(Lx - x^2)(Ly - y^2)\,\mathrm{e}^{-|x-y|/L} \tag{52}$$

with boundary conditions

$$C = 0, \quad \text{at} \quad x = 0, L \quad \text{or} \quad y = 0, L. \tag{53}$$

The complementary solution of eqn (52) is the same as that in eqn (32). The particular solution reads:

$$\begin{aligned}
\phi(u,v) = \frac{q^2}{4\bar{D}_{1t}}[&(8L+3u)(4L-v)L^6 - \tfrac{2}{3}(L+u)L^4v^3 + \tfrac{1}{6}(8L+3u)L^3v^4 \\
&- \tfrac{1}{10}(7L+2u)L^2v^5 + \tfrac{1}{15}(2L+u)Lv^6 - \tfrac{1}{21}Lv^7 \\
&+ L^4(4L^2+3Lu+u^2)(8L^2-5Lv+v^2)\,\mathrm{e}^{-(u-v)/L} \\
&- L^5(8L+3v)(4L^2+3Lu+u^2)\,\mathrm{e}^{-u/L} \\
&- L^5(8L+3u)(4L^2+3Lv+v^2)\,\mathrm{e}^{-v/L}]
\end{aligned} \tag{54}$$

for $x \geqslant y$. The boundary conditions require

$$\left.\begin{aligned}
\mathrm{f}_1(0) + y\mathrm{f}_2(0) + \mathrm{g}_1(y) + \phi(y,0) &= 0 \\
\mathrm{f}_1(L) + y\mathrm{f}_2(L) + \mathrm{g}_1(L) + L\mathrm{g}_2(y) + \phi(L,y) &= 0 \\
\mathrm{f}_1(x) + \mathrm{g}_1(0) + x\mathrm{g}_2(0) + \phi(x,0) &= 0 \\
\mathrm{f}_1(x) + L\mathrm{f}_2(x) + \mathrm{g}_1(L) + x\mathrm{g}_2(L) + \phi(L,x) &= 0.
\end{aligned}\right\} \tag{55}$$

By solving for $\mathrm{f}_1(x)$, $\mathrm{f}_2(x)$, $\mathrm{g}_1(y)$, and $\mathrm{g}_2(x)$ and noting that $\phi(x,0) = \phi(y,0) = 0$, the solution of covariance function becomes

$$C(x,y) = \phi(x,y) + \frac{xy}{L^2}\phi(L,L) - \frac{1}{L}[x\phi(L,y) + y\phi(L,x)], \quad \text{for} \quad x \geqslant y \tag{56}$$

$$C(x,y) = \phi(y,x) + \frac{xy}{L^2}\phi(L,L) - \frac{1}{L}[x\phi(L,y) + y\phi(L,x)], \quad \text{for} \quad x \leqslant y. \tag{57}$$

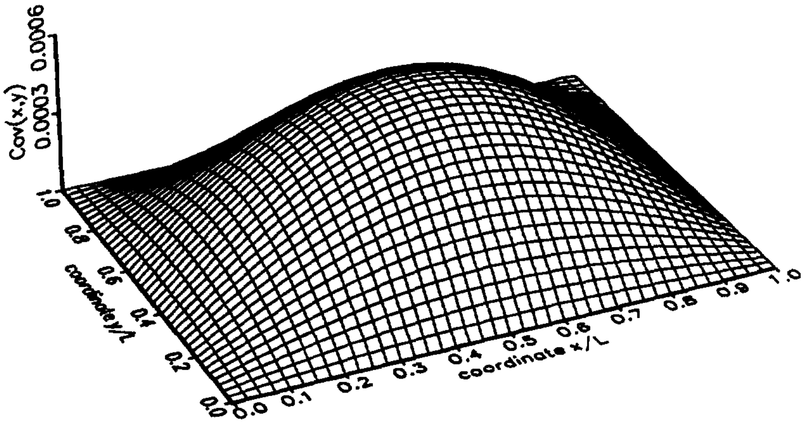

Fig. 5. Covariance function $C(x, y)$ of the displacement for a simply supported beam under uniform pressure, normalized by $q^2/4\bar{D}_{1r}$.

Figure 5 shows the calculated covariance function $C(x, y)$, normalized by the constant $q^2/4\bar{D}_{1r}$. For this simply-supported beam, $C(x, y)$ reaches its maximum at $x = y = L/2$, since the maximum displacement occurs at the mid-point of the beam.

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APPENDIX

Proof of boundary conditions

To prove the boundary conditions of eqns (10)–(12), we just consider cases of free and clamped ends. The appropriate combination of boundary conditions for these two cases will yield other boundary conditions, such as those pertinent to simple supports.

Assume first that the beam is clamped at end $x = 0$. The boundary conditions for the displacement $w(x)$ are

$$w(0) = 0, \quad w'(0) = \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{x=0} = 0. \tag{A1}$$

By taking the expectation operator and noting that the expectation operator and differential operator are interchangeable, we obtain:

$$\bar{w}(0) = 0, \quad \bar{w}'(0) = \left.\frac{\mathrm{d}\bar{w}}{\mathrm{d}x}\right|_{x=0} = 0. \tag{A2}$$

Then, for an arbitrary displacement $w(y)$, we obtain:

$$\left.\begin{aligned} C(0,y) &= \mathrm{E}\{[w(0)-\bar{w}(0)][w(y)-\bar{w}(y)]\} = 0 \\ C'_x(0,y) &= \frac{\partial}{\partial x}\mathrm{E}\{[w(x)-\bar{w}(x)][w(y)-\bar{w}(y)]\}|_{x=0} \\ &= \mathrm{E}\left\{\left[\frac{\mathrm{d}w(x)}{\mathrm{d}x}-\frac{\mathrm{d}\bar{w}(x)}{\mathrm{d}x}\right][w(y)-\bar{w}(y)]\right\}\bigg|_{x=0} \\ &= \mathrm{E}\{[w'(0)-\bar{w}'(0)][w(y)-\bar{w}(y)]\} = 0 \end{aligned}\right\}. \tag{A3}$$

Similarly,

$$\begin{aligned} C(x,0) &= 0 \\ C'_y(x,0) &= 0. \end{aligned} \tag{A4}$$

Assume now that the beam is free at end $x = 0$ but subjected to a prescribed moment $\bar{M}$ and concentrated force $\bar{Q}$. The boundary conditions for displacement $w(x)$ are

$$EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2} = \bar{M}, \quad \frac{\mathrm{d}}{\mathrm{d}x}\left[EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right] = \bar{Q}. \tag{A5}$$

We rewrite the first condition and take expectation to obtain immediately

$$D_0\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}x^2} = \bar{M} \tag{A6}$$

and

$$\frac{\mathrm{d}^2[w-\bar{w}]}{\mathrm{d}x^2} = \bar{M}\left[\frac{1}{EI}-\frac{1}{D_0}\right]. \tag{A7}$$

Substracting eqn (4) from eqn (2) and then multiplying by eqn (A7), we obtain

$$\frac{\mathrm{d}^2[w(y)-\bar{w}(y)]}{\mathrm{d}y^2}\frac{\mathrm{d}^2[w-\bar{w}]}{\mathrm{d}x^2} = m(y)\bar{M}\left[\frac{1}{EI}-\frac{1}{D_0}\right]\left[\frac{1}{EI(y)}-\frac{1}{D_0(y)}\right]. \tag{A8}$$

By taking expectation, we obtain

$$D_1(0,y)\frac{\partial^4 C}{\partial x^2\,\partial y^2}\bigg|_{x=0} = m(y)\bar{M}. \tag{A9}$$

Similarly

$$D_1(x,0)\frac{\partial^4 C}{\partial x^2\,\partial y^2}\bigg|_{y=0} = m(x)\bar{M}. \tag{A10}$$

Rewriting eqn (2) and bearing in mind that $M_0 = \bar{M}$ and $Q_0 = \bar{Q}$ for free ends, we obtain

$$\frac{\mathrm{d}^2w}{\mathrm{d}x^2} = \frac{1}{EI(x)}\left[\bar{Q}x+\bar{M}-\int_0^x\int_0^v q(u)\,\mathrm{d}u\,\mathrm{d}v\right]. \tag{A11}$$

Taking expectation of eqn (A11), multiplying it by D_0 and differentiating the result and then letting $x = 0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[D_0\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}x^2}\right]\bigg|_{x=0} = \bar{Q}. \tag{A12}$$

Taking expectation of eqn (7), multiplying by D_1, and differentiating the result, and then letting $x = 0$ yield

$$\frac{\partial}{\partial x}\left[D_1\frac{\partial^4 C}{\partial x^2\partial y^2}\right]\bigg|_{x=0} = m(y)\bar{Q}. \tag{A13}$$

Analogously, the boundary condition at $y = 0$ reads

$$\frac{\partial}{\partial y}\left[D_1\frac{\partial^4 C}{\partial x^2\partial y^2}\right]\bigg|_{y=0} = m(x)\bar{Q}. \tag{A14}$$